Primitive edge rotation groups of weighted trees

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Dessins d'enfant

$D = (X, \Gamma)$ – dessin d'enfant (*bicolored* map):

- X – compact connected oriented surface without boundary;
- Γ – bicolored graph, $\Gamma \subset X$;
- $X \setminus \Gamma =$ disjoint union of discs.

$(\mathcal{X}, \beta)$ – Belyi pair:

- $\mathcal{X}$ – complex algebraic curve over $\overline{\mathbf{Q}}$;
- β – rational function on $\mathcal{X}$, β is ramified over 0, 1 and ∞ only (Belyi function).

$\Gamma = \beta^{-1}[0, 1]$

Galois action on dessins

$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$: Belyi pairs $\{(\mathcal{X}, \beta)\}$

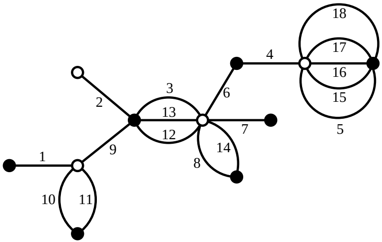
$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$: dessins d'enfant $\{(X, \Gamma)\}$

Invariants of this action:

- passport
- symmetries $\mathrm{Aut}(D)$
- invariance under color exchange, self-duality
- edge rotation group $ER(D)$

Edge rotation group $ER(D)$

$D = (X, \Gamma)$



$$
\begin{aligned}
a &= (2,9,12,13,3)(4,6)(5,18,17,16,15)(8,14)(10,11),\\
b &= (1,10,11,9)(3,13,12,8,14,7,6)(4,5,15,16,17,18),\\
c &= (1,2,3,4,5,6,7,8,9,10).
\end{aligned}
$$

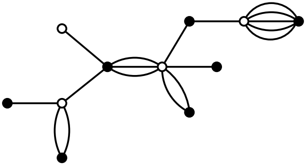
$ER(D) = \langle a, b \rangle \subseteq \mathbf{S}_n$, n – number of edges

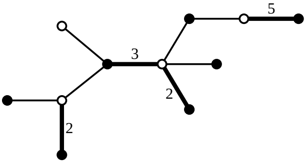
$(\mathcal{X}, \beta)$
$ER(D) \simeq Mon(\beta)$

Weighted trees: definition

Weighted tree is a map of genus 0 such that all its faces except one has degree 1.

Degrees of faces of a weidghted tree are $(1^r(n-r)^1)$.



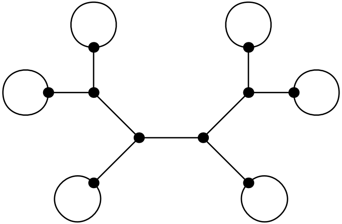


Motivation: $P^3 - Q^2$ (1965, B. J. Birch et al.)

What is the minimum degree of $P^3 - Q^2$?

Let $R = P^3 - Q^2$ and $f = \dfrac{P^3}{R} = \dfrac{Q^2}{R} + 1$.

The minimum degree of R is attained when f is a Belyi function of a weighted tree.



Motivation: Davenport's bound

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_p), \qquad \beta = (\beta_1, \beta_2, \ldots \beta_q), \qquad \sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = n,$$

$$P(x) = \prod_{i=1}^{p} (x - a_i)^{\alpha_i}, \qquad Q(x) = \prod_{j=1}^{q} (x - b_j)^{\beta_j}, \quad a_i \text{ and } b_j \text{ are all distinct.}$$

- $\gcd(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q) = 1$
- $p + q \leq n + 1$

Theorem (U. Zannier, 1995)

1. $\deg(P - Q) \geq (n + 1) - (p + q)$.
2. This bound is always attained, whatever are α and β.

Motivation: plane trees of diameter 4

N. Adrianov, G. Shabat (1990): Trees of diameter 4

Anti-Wandermonde system

$$\left\{ \sum_{i=1}^{p} a_i x_i^k = 0, \qquad k = 1, \ldots p-1 \right.$$

L. Schneps: splitting orbits $IV[1,2,3,4,6]$ and $IV[2,3,4,5,6]$

Yu. Kochetkov (conjecture): $IV[a_1,a_2,a_3,a_4,a_5]$ splits when

$$D = a_1 a_2 a_3 a_4 a_5 (a_1 + a_2 + a_3 + a_4 + a_5) \quad \text{is a full square}$$

L. Zapponi: $IV[a_1,a_2,a_3,a_4,a_5]$ always splits over $\mathbf{Q}(\sqrt{D})$

Motivation: plane trees of diameter 4

Yu. Kochetkov (2013):

1. Trees of diameter 4 are *dual* to weighted trees

2. Anti-Wandermonde system

$$\left\{\sum_{i=1}^{p} a_i x_i^k = \sum_{j=1}^{q} b_j y_j^k, \qquad k = 1, \dots p+q-2\right.$$

3. The orbit splits over $\mathbf{Q}(\sqrt{a_1 \dots a_p \cdot b_1 \dots b_q})$

4. Enumeration of weighted trees by passport:

$$(p+q-2)! \qquad \text{in many cases}$$

Which groups can appear as the edge rotation groups of weighted trees?

Theorem (W. Feit, 1980)

Let $G \subseteq \mathbf{S}_n$ be a primitive permutation group which contains n-cycle. Then

(i) $G = \mathbf{A}_n$ or $\mathbf{S}_n$;

(ii) $n = p$ – prime and $\mathbf{C}_p \subseteq G \subseteq \mathbf{AGL}_1(p)$;

(iii) $n = 11$ and $G = \mathbf{PSL}_2(11)$ or $\mathbf{M}_{11}$;

(iv) $n = 23$ and $G = \mathbf{M}_{23}$;

(v) $n = (q^d - 1)/(q - 1)$ and $\mathbf{PSL}_d(q) \subseteq G \subseteq \mathbf{P\Gamma L}_d(q)$.

(based on the classification of finite simple groups)

Theorem (P. Müller, 1995)

Primitive monodromy groups of polynomials are classified.

Most of these polynomials have 2 critical values (Shabat polynomials) and can be drawn as plane trees.

3 exceptional cases with polynomials having 3 critical values:

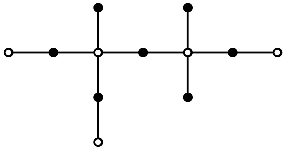
- $G = \mathbf{PSL}_3(2)$ with $n = 7$ and passport $(1^3 2^2, 1^3 2^2, 1^3 2^2, 7^1)$;
- $G = \mathbf{PSL}_3(3)$ with $n = 13$ and passport $(1^5 2^4, 1^5 2^4, 1^5 2^4, 13^1)$;
- $G = \mathbf{PSL}_4(2)$ with $n = 15$ and passport $(1^3 2^6, 1^7 2^4, 1^7 2^4, 15^1)$.

N. Adrianov, Yu. Kochetkov, A. Suvorov, 1997

A complete list of special plane trees is obtained.

(T is special if $ER(T)$ is primitive and
$ER(T) \neq \mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{A}_n$, $\mathbf{S}_n$.)

Mathieu group $\mathbf{M}_{11}$:



Dessins are probably the easiest way to define Mathieu groups!

Theorem (G. Jones, 2012)

Let $G \subseteq \mathbf{S}_n$ be a primitive permutation group, $G \neq \mathbf{A}_n$, $\mathbf{S}_n$, and G contains a permutation of type $(1^r, n-r)$. Then

1. $r = 0$
   - (i) $\mathbf{C}_p \subseteq G \subseteq \mathbf{AGL}_1(p)$ with $n = p$ prime;
   - (ii) $\mathbf{PGL}_d(q) \subseteq G \subseteq \mathbf{P\Gamma L}_d(q)$ with $n = (q^d - 1)/(q-1)$, $d \geq 2$;
   - (iii) $G = \mathbf{PSL}_2(11)$, $\mathbf{M}_{11}$ or $\mathbf{M}_{23}$ with $e = 11$, 11 or 23.
2. $r = 1$
   - (i) $\mathbf{AGL}_d(q) \subseteq G \subseteq \mathbf{A\Gamma L}_d(q)$ with $n = q^d$ and $d \geq 1$;
   - (ii) $G = \mathbf{PSL}_2(p)$ or $\mathbf{PGL}_2(p)$ with $n = p + 1$ for prime $p \geq 5$;
   - (iii) $G = \mathbf{M}_{11}$, $\mathbf{M}_{12}$ or $\mathbf{M}_{24}$ with $e = 12$, 12 or 24.
3. $r = 2$

   $\mathbf{PGL}_2(q) \subseteq G \subseteq \mathbf{P\Gamma L}_2(q)$ with $n = q + 1$.

Theorem (N. Adrianov, A.Zvonkin, 2014)

A complete list of special weighted trees is obtained.

- 34 different edge rotation groups;
- 184 weighted trees (up to color exchange);
- subdivided into (at least) 85 Galois orbits.

GAP – Groups, Algorithms, Programming

a System for Computational Discrete Algebra
`http://www.gap-system.org/`

- Primitive permutation groups (any degree < 2500)
- Character tables of finite groups

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1. Find generators:
   - Loop over all primitive groups G of given degree
   - Loop over all triples of conjugacy classes (A, B, C) of G which may give a weighted tree
   - Fix an element in a largest class $a \in A$, loop over all elements in a smallest class $b \in B$, to get $(ab)^{-1} \in C$
2. Draw weighted trees (generate Metapost code)

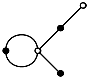
5	$\mathbf{AGL}_1(5)$	$(1^1 2^2, 1^1 4^1, 1^1 4^1)$	2 (1)
6	$\mathbf{PSL}_2(5)$	$(1^2 2^2, 3^2, 1^1 5^1)$	1
		$(1^2 2^2, 1^1 5^1, 1^1 5^1)$	1
6	$\mathbf{PGL}_2(5)$	$(1^2 2^2, 1^2 4^1, 6^1)$	1
		$(2^3, 1^2 4^1, 1^1 5^1)$	1
		$(2^3, 1^1 5^1, 1^2 4^1)$	1
		$(1^2 2^2, 6^1, 1^2 4^1)$	1
		$(3^2, 1^2 4^1, 1^2 4^1)$	1
		$(1^2 4^1, 1^2 4^1, 1^1 5^1)$	1
		$(1^2 4^1, 1^1 5^1, 1^2 4^1)$	1
7	$\mathbf{AGL}_1(7)$	$(1^1 2^3, 1^1 3^2, 1^1 6^1)$	2(1)
7	$\mathbf{PSL}_3(2)$	$(1^3 2^2, 1^1 3^2, 7^1)$	2(1)
		$(1^3 2^2, 1^1 2^1 4^1, 7^1)$	2(1)
8	$\mathbf{A\Gamma L}_1(8)$	$(2^4, 1^2 3^2, 1^1 7^1)$	2(1)
8	$\mathbf{ASL}_3(2)$	$(2^4, 1^2 2^1 4^1, 1^1 7^1)$	2(1)
		$(1^4 2^2, 4^2, 1^1 7^1)$	2(1)
		$(1^4 2^2, 2^1 6^1, 1^1 7^1)$	2(1)
		$(1^4 2^2, 1^1 7^1, 1^1 7^1)$	2(1)
		$(1^2 3^2, 1^2 2^1 4^1, 1^1 7^1)$	4(2)
		$(1^2 2^1 4^1, 1^2 2^1 4^1, 1^1 7^1)$	2(1)
8	$\mathbf{PSL}_2(7)$	$(2^4, 1^2 3^2, 1^1 7^1)$	1
		$(1^2 3^2, 1^2 3^2, 1^1 7^1)$	1

8	$\mathbf{PGL}_2(7)$	$(1^2 2^3, 4^2, 1^2 6^1)$	1
		$(2^4, 1^2 6^1, 1^2 6^1)$	1
		$(1^2 2^3, 1^2 6^1, 1^1 7^1)$	1
		$(1^2 2^3, 1^1 7^1, 1^2 6^1)$	1
		$(1^2 3^2, 1^2 6^1, 1^2 6^1)$	1
		$(1^2 2^3, 1^2 3^2, 8^1)$	2
9	$\mathbf{A\Gamma L}_1(9)$	$(1^3 2^3, 1^1 4^2, 1^1 8^1)$	2(1)
9	$\mathbf{AGL}_2(3)$	$(1^3 2^3, 3^3, 1^1 8^1)$	2(1)
		$(1^3 2^3, 1^1 2^1 6^1, 1^1 8^1)$	2(1)
9	$\mathbf{PSL}_2(8)$	$(1^1 2^4, 3^3, 1^2 7^1)$	1
		$(1^1 2^4, 1^2 7^1, 1^2 7^1)$	2
9	$\mathbf{P\Gamma L}_2(8)$	$(1^1 2^4, 1^3 3^2, 9^1)$	2(1)
		$(1^3 3^2, 1^3 3^2, 9^1)$	4(2)
		$(1^3 3^2, 1^1 2^1 6^1, 1^2 7^1)$	4(2)
10	$\mathbf{PGL}_2(9)$	$(2^5, 1^1 3^3, 1^2 8^1)$	1
		$(2^5, 1^2 4^2, 1^2 8^1)$	1
		$(1^2 2^4, 1^2 8^1, 1^2 8^1)$	1
10	$\mathbf{P\Gamma L}_2(9)$	$(1^4 2^3, 2^1 8^1, 1^2 8^1)$	1
		$(1^4 2^3, 1^2 4^2, 10^1)$	1
11	$\mathbf{PSL}_2(11)$	$(1^3 2^4, 1^2 3^3, 11^1)$	2(1)
11	$\mathbf{M}_{11}$	$(1^3 2^4, 1^3 4^2, 11^1)$	2(1)

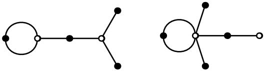
12	$\mathbf{M}_{11}$	$(1^4 2^4, 2^2 4^2, 1^1 11^1)$	2(1)
		$(1^4 2^4, 1^2 5^2, 1^1 11^1)$	2(1)
		$(1^4 2^4, 1^1 2^1 3^1 6^1, 1^1 11^1)$	6(3)
12	$\mathbf{M}_{12}$	$(1^4 2^4, 3^4, 1^1 11^1)$	2(1)
		$(2^6, 1^3 3^3, 1^1 11^1)$	2(1)
		$(2^6, 1^4 4^2, 1^1 11^1)$	2(1)
		$(1^4 2^4, 1^2 5^2, 1^1 11^1)$	2(1)
		$(1^4 2^4, 1^1 2^1 3^1 6^1, 1^1 11^1)$	2(1)
		$(1^3 3^3, 1^3 3^3, 1^1 11^1)$	2(1)
		$(1^4 2^4, 1^2 2^1 8^1, 1^1 11^1)$	4(2)
		$(1^3 3^3, 1^4 4^2, 1^1 11^1)$	2(1)
		$(1^4 4^2, 1^4 4^2, 1^1 11^1)$	2(1)
12	$\mathbf{PGL}_2(11)$	$(1^2 2^5, 3^4, 1^2 10^1)$	2
		$(1^2 2^5, 1^2 5^2, 1^2 10^1)$	2
13	$\mathbf{PSL}_3(3)$	$(1^5 2^4, 1^1 3^4, 13^1)$	4(2)
		$(1^5 2^4, 1^1 2^2 4^2, 13^1)$	4(2)
		$(1^5 2^4, 1^2 2^1 3^1 6^1, 13^1)$	4(2)
14	$\mathbf{PSL}_2(13)$	$(1^2 2^6, 1^2 3^4, 1^1 13^1)$	1
14	$\mathbf{PGL}_2(13)$	$(2^7, 1^2 3^4, 1^2 12^1)$	2
		$(1^2 2^6, 1^2 4^3, 1^2 12^1)$	2
15	$\mathbf{PSL}_4(2)$	$(1^3 2^6, 1^3 2^2 4^2, 15^1)$	2(1)
		$(1^7 2^4, 1^1 2^1 4^3, 15^1)$	2(1)
		$(1^7 2^4, 1^1 2^1 3^2 6^1, 15^1)$	2(1)

16	$2^4.\mathbf{PSL}_4(2)$	$(1^4 2^6, 2^4 4^2, 1^1 15^1)$	2(1)
		$(1^8 2^4, 1^1 5^3, 1^1 15^1)$	2(1)
		$(1^8 2^4, 1^1 3^1 6^2, 1^1 15^1)$	2(1)
		$(1^4 2^6, 1^2 2^1 3^2 6^1, 1^1 15^1)$	6(3)
16	$\mathbf{A\Gamma L}_2(4)$	$(1^4 2^6, 1^2 2^1 4^3, 1^1 15^1)$	2(1)
17	$\mathbf{PSL}_2(16)$	$(1^1 2^8, 1^2 3^5, 1^2 15^1)$	1
17	$\mathbf{PSL}_2(16){:}2$	$(1^5 2^6, 1^1 4^4, 1^2 15^1)$	1
20	$\mathbf{PGL}_2(19)$	$(1^2 2^9, 1^2 3^6, 1^2 18^1)$	3
21	$\mathbf{P\Gamma L}(3,4)$	$(1^7 2^7, 1^3 2^1 4^4, 21^1)$	2(1)
23	$\mathbf{M}_{23}$	$(1^7 2^8, 1^3 2^2 4^4, 23^1)$	4(2)
24	$\mathbf{M}_{24}$	$(2^{12}, 1^6 3^6, 1^1 23^1)$	2(1)
		$(1^8 2^8, 3^8, 1^1 23^1)$	2(1)
		$(1^8 2^8, 1^4 5^4, 1^1 23^1)$	2(1)
		$(1^6 3^6, 1^6 3^6, 1^1 23^1)$	2(1)
		$(1^8 2^8, 1^2 2^2 3^2 6^2, 1^1 23^1)$	10(5)
31	$\mathbf{PSL}_5(2)$	$(1^7 2^{12}, 1^3 2^6 4^4, 31^1)$	6(3)
32	$\mathbf{ASL}_5(2)$	$(1^8 2^{12}, 1^2 3^{10}, 1^1 31^1)$	6(3)

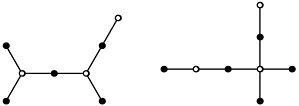
5.1. $\mathbf{AGL}_1(5)$



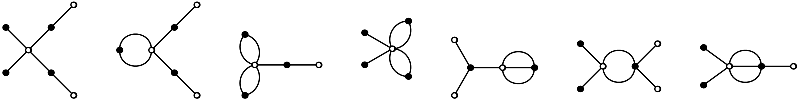
6.1 and 6.2. $\mathbf{PSL}_2(5)$



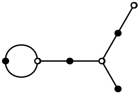
7.2 and 7.3. $\mathbf{PSL}_3(2)$



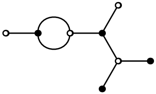
6.3 – 6.9. $\mathbf{PGL}_2(5)$



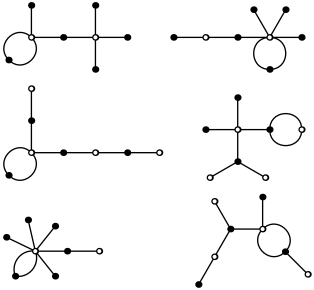
7.1. $\mathbf{AGL}_1(7)$



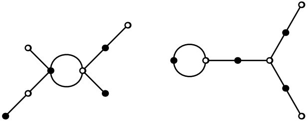
8.1. $\mathbf{A\Gamma L}_1(8)$



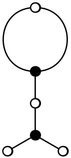
8.2 – 8.7. $\mathbf{ASL}_3(2)$



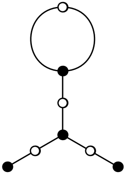
8.8 – 8.9. $\mathbf{PSL}_2(7)$



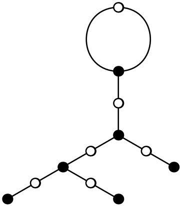
“Funny family”



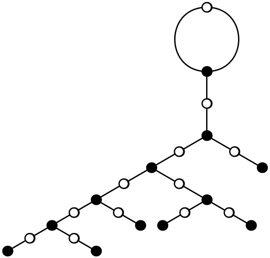
baby: $\mathbf{PSL}_2(5)$



child: $\mathbf{PSL}_2(7)$

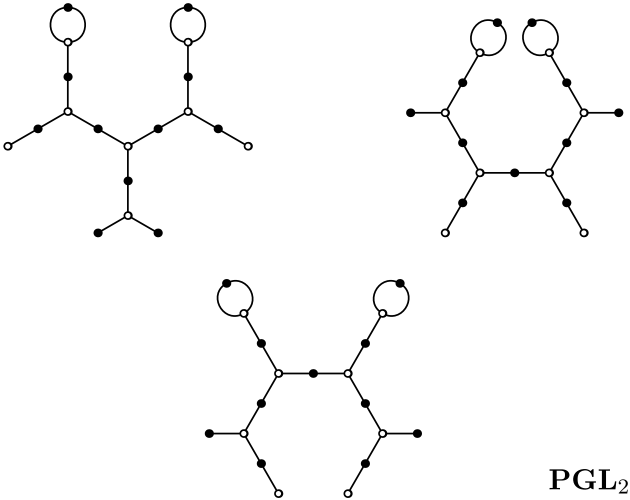


teen: $\mathbf{M}_{12}$



adult: $\mathbf{M}_{24}$

“TWINS”



$\mathbf{PGL}_2(19)$

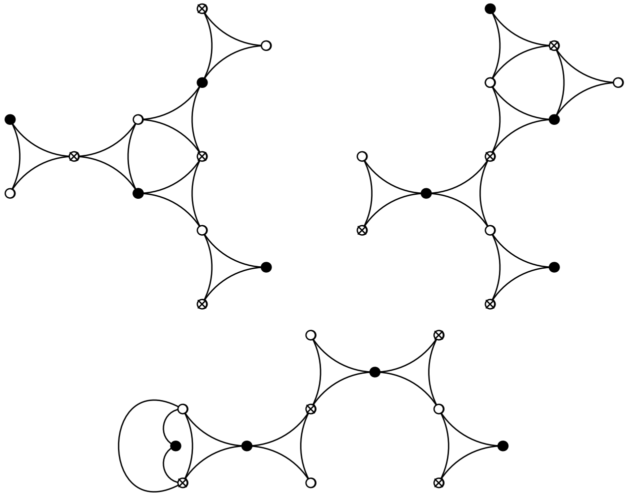
More than 3 critical values?

D is a weighted tree $\iff$ β_D has only one multiple pole.

G.Jones' theorem lets us classify primitive monodromy groups of rational functions with only one multiple pole

4/6.1	$\mathbf{PSL}_2(5)$	$(1^2 2^2, 1^2 2^2, 1^2 2^2, 1^1 5^1)$	1×10
4/6.2	$\mathbf{PGL}_2(5)$	$(1^2 2^2, 1^2 2^2, 2^3, 1^2 4^1)$	1×8
4/6.3		$(1^2 2^2, 1^2 2^2, 1^2 4^1, 1^2 4^1)$	1×16
4/7.1	$\mathbf{PSL}_3(2)$	$(1^3 2^2, 1^3 2^2, 1^3 2^2, 7^1)$	2×7
5/8.1	$\mathbf{ASL}_3(2)$	$(1^4 2^2, 1^4 2^2, 1^4 2^2, 1^4 2^2, 1^1 7^1)$	2×147
4/8.1		$(1^4 2^2, 1^4 2^2, 2^4, 1^1 7^1)$	2×7
4/8.2		$(1^4 2^2, 1^4 2^2, 1^2 2^1 4^1, 1^1 7^1)$	2×14
4/8.3		$(1^4 2^2, 1^4 2^2, 1^2 3^2, 1^1 7^1)$	2×21
4/8.4	$\mathbf{PGL}_2(7)$	$(1^2 2^3, 1^2 2^3, 1^2 2^3, 1^2 6^1)$	1×18
4/9.1	$\mathbf{AGL}_2(3)$	$(1^3 2^3, 1^3 2^3, 1^3 2^3, 1^1 8^1)$	2×16
4/10.1	$\mathbf{P\Gamma L}_2(9)$	$(1^4 2^3, 1^4 2^3, 2^5, 1^2 8^1)$	1×8
4/12.1	$\mathbf{M}_{11}$	$(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$	2×33
4/12.2	$\mathbf{M}_{12}$	$(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$	2×22
4/13.1	$\mathbf{PSL}_3(3)$	$(1^5 2^4, 1^5 2^4, 1^5 2^4, 13^1)$	4×13
4/15.1	$\mathbf{PSL}_4(2)$	$(1^7 2^4, 1^7 2^4, 1^3 2^6, 15^1)$	2×5
4/16.1	$\mathbf{AGL}_4(2)$	$(1^8 2^4, 1^4 2^6, 1^4 2^6, 1^1 15^1)$	2×15
4/24.1	$\mathbf{M}_{24}$	$(1^8 2^8, 1^8 2^8, 1^8 2^8, 1^1 23^1)$	2×46

$4/6.1$: $\mathbf{PSL}_2(5)$



$$10 = 1 + 3 + 3 \cdot 2$$

Braid group action

Braid group

$$B_k = \langle \sigma_1, \sigma_2, \ldots \sigma_{k-1} \quad | \quad \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \ \text{ for } \ |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle \end{array}$$

Hurwitz (spherical) braid group

$$H_k = \langle \sigma_1, \sigma_2, \ldots \sigma_{k-1} \quad | \quad \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \ \text{ for } \ |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_1\sigma_2 \ldots \sigma_{k-2}\sigma_{k-1}^2\sigma_{k-2} \ldots \sigma_2\sigma_1 = 1 \rangle \end{array}$$

$$\sigma_i : (g_1, g_2, \ldots g_k) \longrightarrow (g_1, g_2, \ldots g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \ldots g_k)$$

$$\begin{array}{rcl} \sigma_i(g_i) & = & g_{i+1} \\ \sigma_i(g_{i+1}) & = & g_{i+1}^{-1} g_i g_{i+1} \\ \sigma_i(g_j) & = & g_j, \quad \text{for} \quad j \neq i, i+1. \end{array}$$

Megamaps (A. Zvonkin)

(C_1, C_2, C_3, C_4) – quadruple of conjugacy classes of G

$$\mathcal{G} = \{(g_1, g_2, g_3, g_4) \mid g_i \in C_i,\, g_1 g_2 g_3 g_4 = 1\}/\sim$$

$$\Sigma = \sigma_1^2$$
$$\mathrm{A} = \sigma_2^2$$
$$\Phi = \sigma_2^{-1}\sigma_3^2\sigma_2$$

$$\Sigma\mathrm{A}\Phi = \sigma_1^2 \cdot \sigma_2^2 \cdot \sigma_2^{-1}\sigma_3^2\sigma_2 = \sigma_1(\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1)\sigma_1^{-1} = 1 \quad \text{in } H_4$$

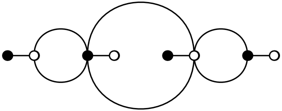
Megamap is a dessin defined by action of (Σ, A) on $\mathcal{G}$.

4/6.1: $\mathbf{PSL}_2(5)$ $(1^2 2^2, 1^2 2^2, 1^2 2^2, 1^1 5^1)$ 1×10

$g = 0$

$(1^2 3^1 5^1, 1^2 3^1 5^1, 1^2 3^1 5^1)$

$ER = \mathbf{A}_{10}$

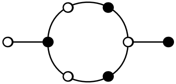


4/6.2: $\mathbf{PGL}_2(5)$ $(1^2 2^2, 1^2 2^2, 2^3, 1^2 4^1)$ 1×8

$g = 0$

$(1^1 2^2 3^1, 3^1 5^1, 1^1 2^2 3^1)$

$ER = \mathbf{A}_8$

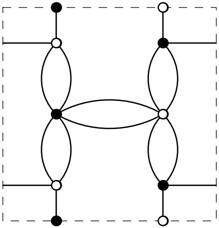


4/6.3: $\mathbf{PGL}_2(5)$ $(1^2 2^2, 1^2 2^2, 1^2 4^1, 1^2 4^1)$ 1×16

$g = 1$

$(1^5 3^2 5^1, 2^1 4^2 6^1, 2^1 4^2 6^1)$

$ER =$ imprimitive group of order $812\,851\,200$



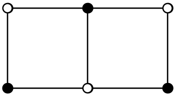
4/7.1: $\mathbf{PSL}_3(2)$ $(1^3 2^2, 1^3 2^2, 1^3 2^2, 7^1)$ 2×7

4/8.1: $\mathbf{ASL}_3(2)$ $(1^4 2^2, 1^4 2^2, 2^4, 1^1 7^1)$ 2×7

$g = 0$

$(2^2 3^1, 2^2 3^1, 2^2 3^1)$

$ER = \mathbf{A}_7$

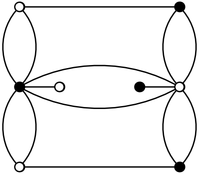


4/8.2: $\mathbf{ASL}_3(2)$ $(1^4 2^2, 1^4 2^2, 1^2 2^1 4^1, 1^1 7^1)$ 2×14

$g = 0$

$(1^4 2^2 3^2, 1^1 3^2 7^1, 1^1 3^2 7^1)$

$ER = \mathbf{A}_{14}$

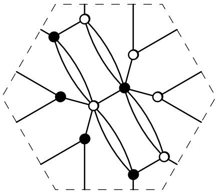


4/8.3: $\mathbf{ASL}_3(2)$ $(1^4 2^2, 1^4 2^2, 1^2 3^2, 1^1 7^1)$ 2×21

$g = 1$

$(3^2 4^2 7^1, 1^4 2^4 3^3, 3^2 4^2 7^1)$

$ER = \mathbf{A}_{21}$

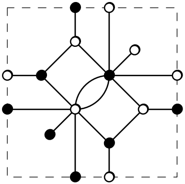


4/8.4: $\mathbf{PGL}_2(7)$ $(1^2 2^3, 1^2 2^3, 1^2 2^3, 1^2 6^1)$ 1×18

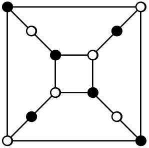
$g = 1$

$(1^1 2^2 3^2 7^1, 1^1 2^2 3^2 7^1, 1^1 2^2 3^2 7^1)$

$ER = \mathbf{A}_{18}$



4/9.1: $\mathbf{AGL}_2(3)$ $(1^3 2^3, 1^3 2^3, 1^3 2^3, 1^1 8^1)$ 2×16

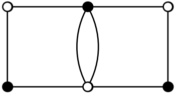


$g = 0$

$(2^2 3^4, 2^2 3^4, 2^2 3^4)$

$ER =$ imprimitive group of order 3072

4/10.1: $\mathbf{P\Gamma L}_2(9)$ $(1^4 2^3, 1^4 2^3, 2^5, 1^2 8^1)$ 1×8



$g = 0$

$(1^1 2^2 3^1, 2^2 4^1, 2^2 4^1)$

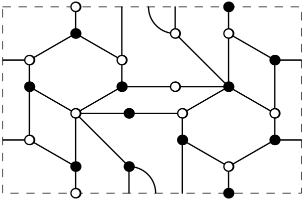
$ER =$ imprimitive group of order 576

4/12.1: $\mathbf{M}_{11}$ $(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$ 2×33

$g = 1$

$(2^1 3^8 5^1, 2^1 3^8 5^1, 2^1 3^8 5^1)$

$ER = \mathbf{A}_{33}$

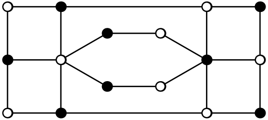


4/12.2: $\mathbf{M}_{12}$ $(1^4 2^4, 1^4 2^4, 1^4 2^4, 1^1 11^1)$ 2×22

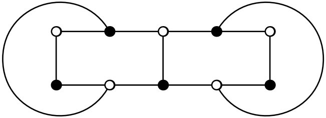
$g = 0$

$(2^4 3^3 5^1, 2^4 3^3 5^1, 2^4 3^3 5^1)$

$ER = \mathbf{A}_{22}$



4/13.1: $\mathbf{PSL}_3(3)$ $(1^5 2^4, 1^5 2^4, 1^5 2^4, 13^1)$ 4×13

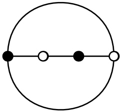


$g = 0$

$(2^2 3^3, 2^2 3^3, 2^2 3^3)$

$ER = \mathbf{A}_{13}$

4/15.1: $\mathbf{PSL}_4(2)$ $(1^7 2^4, 1^7 2^4, 1^3 2^6, 15^1)$ 2×5



$g = 0$

$(2^1 3^1, 2^1 3^1, 1^1 2^2)$

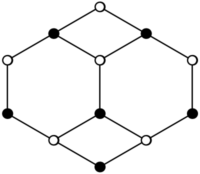
$ER = \mathbf{S}_5$

4/16.1: $\mathbf{AGL}_4(2)$ $(1^8 2^4, 1^4 2^6, 1^4 2^6, 1^1 15^1)$ 2×15

$g = 0$

$(2^3 3^3, 2^3 3^3, 2^2 3^2 5^1)$

$ER = \mathbf{S}_{15}$



4/24.1: $\mathbf{M}_{24}$ $(1^8 2^8, 1^8 2^8, 1^8 2^8, 1^1 23^1)$ 2×46

$g = 0$

$(2^4 3^{11} 5^1, 2^4 3^{11} 5^1, 2^4 3^{11} 5^1)$

$ER = \mathbf{A}_{46}$